A Lighting-Invariant Point Processor for Shading: 
Supplementary Material

Kathryn Heal Jialiang Wang Steven Gottler Todd Zickler
Harvard University

{kathrynheal@g, jialiangwang@g, sjg@cs, zickler@seas}.harvard.edu

1. \( \tilde{I}_{yy} \leq 0 \) from §4.3.

Lemma 1. The rotation-induced transformation \( R_1 \) that maps \( I_y \mapsto 0 \) always maps \( I_{yy} \) to a nonpositive value.

Proof. Under this transformation,

\[
R_1 : I = \left( \begin{array}{c} I \\ I_x \\ I_y \\ I_{xx} \\ I_{xy} \\ I_{yy} \end{array} \right) \mapsto \tilde{I} = \left( \begin{array}{c} I \\ +\sqrt{I_x^2 + I_y^2} \\ 0 \\ I_x^2 I_{xx} + 2 I_x I_{xy} I_y + I_y^2 I_{yy} \\ I_x^2 I_{xy} + I_y (I_{yy} - I_{xx}) - I_{xx} I_{yy} \\ I_y^2 I_{yy} - 2 I_x I_{xy} I_y + I_{xx} I_{yy} \end{array} \right) =: \left( \begin{array}{c} \tilde{I} \\ \tilde{I}_x \\ \tilde{I}_y \\ \tilde{I}_{xx} \\ \tilde{I}_{xy} \\ \tilde{I}_{yy} \end{array} \right)
\]

Over \( \mathbb{R} \), \( \tilde{I}_{yy} \) is the same sign as its numerator \( W := I_x^2 I_{yy} - 2 I_x I_{xy} I_y + I_{xx} I_y^2 \), so we’ll study \( \text{sgn}(W) \) as a proxy. Recall that the point processor considers the point \( (x, y) = (0, 0) \). By the quadratic patch assumption the true surface \( f^*(x, y) = ax + by + \frac{1}{2} (c^2 x + 2 d x y + e y^2) \), so at this point we have \( f^* = (a, b, c, d, e)^T \). Recall also that \( I(0, 0) = \rho L \cdot N^*(x, y)/||N^*(x, y)|| \), where \( N^* = \langle -\partial f/\partial x, -(\partial f/\partial y), 1 \rangle = (-a, -b, 1)^T \) and WLOG \( \rho = 1 \). Writing \( I(0, 0) \) as a function of \( (f^*, L^*) \), we have

\[
W = I_x^2 I_{yy} - 2 I_x I_{xy} I_y + I_{xx} I_y^2 = \frac{(d^2 - ce)^2}{(1 + a^2 + b^2)^{9/2} (L_1^2 + L_2^2 + L_3^2)^{3/2}} (aL_1 + bL_2 - L_3) V,
\]

strictly positive

\[
V := L_1^2 + b^2 L_2^2 - 2abL_1L_2 + L_2^2 + a^2 L_3^2 + 2aL_1L_3 + 2bL_2L_3 + a^2 L_3^2 + b^2 L_3^2
\]

\[
= (bL_1 - aL_2)^2 + (L_1 + aL_3)^2 + (L_2 + bL_3)^2
\]

Since \( V \) can be written as the sum of squares, the entirety of \( W \) is nonpositive; therefore, \( \tilde{I}_{yy} \) is always nonpositive. Furthermore, \( V \) can only be zero-valued when \( (a, b) = (-L_1/L_3, -L_2/L_3) \), which generically will not occur.

2. Proof of Theorem 1 from §3.2.

Proof. For brevity let \( a, b, c, d, e = f_x, f_y, f_{xx}, f_{xy}, f_{yy} \). We begin with Lambert’s law for some fixed \( I \) vector,

\[
I(x, y) = \rho L \cdot \frac{N(x, y)}{||N(x, y)||}, \quad (x, y) \in U,
\]

(1)
with \( \mathbf{N}(x, y) := (-\partial f/\partial x)(x, y), -(\partial f/\partial y)(x, y), 1)^T = (-a - cx - dy, -b - dx - ey, 1)^T \). Since we do not require \( \mathbf{L} \) to be unit, we can effectively absorb \( \rho \) into it. This turns (1) into

\[
I(x, y) = \frac{-(a + cx + dy)L_1 + (b + dx + ey)L_2 - L_3}{\sqrt{a + cx + dy}^2 + b + dx + ey + 1} = -w((a + cx + dy)L_1 + (b + dx + ey)L_2 - L_3),
\]

with \( w := 1/\sqrt{a + cx + dy}^2 + b + dx + ey + 1 \). Rearranging this,

\[
0 = w((a + cx + dy)L_1 + (b + dx + ey)L_2 - L_3) - I(x, y),
\]

\[
0 = w^3(a + cx + dy)^2 + b + dx + ey + 1 - 1.
\]

Taking first and second order partial derivatives of (3) with respect to \( x \) and \( y \), then evaluating each resulting derivative at the point \( (x, y) = (0, 0) \) gives us the local system

\[
s := w^2(a^2 + b^2 + 1) - 1
\]

\[
r_0 := w(aL_1 + bL_2 - L_3) + I
\]

\[
r_x := w^3(-(ac + bd))(aL_1 + bL_2 - L_3) + w(cL_1 + dL_2) + L_x
\]

\[
r_y := w^3(-(ad + be))(aL_1 + bL_2 - L_3) + w(dL_1 + eL_2) + L_y
\]

\[
r_{xx} := w^6(3w^2(ac + bd)^2 - c^2 - d^2)(aL_1 + bL_2 - L_3) - 2w^3(ac + bd)(cL_1 + dL_2) + I_{xx}
\]

\[
r_{xy} := 3w^5(ac + bd)(ad + be)(aL_1 + bL_2 - L_3) - dw^3(c + e)(aL_1 + bL_2 - L_3)
\]

\[
-w^3(ad + be)(cL_1 + dL_2) - w^3(ac + bd)(dL_1 + eL_2) + I_{xy}
\]

\[
r_{yy} := w^3(3w^2(ac + bd)^2 - d^2 - c^2)(aL_1 + bL_2 - L_3) - 2w^3(ad + be)(dL_1 + eL_2) + I_{yy}
\]

and we refer to the vectors \((r_0, r_x, r_y, r_{xx}), (r_0, r_x, r_y, r_{xy})\), and \((r_0, r_x, r_y, r_{yy})\) as \( r_1, r_2, \) and \( r_3 \), respectively. Each of these seven polynomials is linear in the \( L_i \); thus, if \( \mathbf{L} := (L_1, L_2, L_3, 1) \), we can write this system as \( \mathbf{r}_i = A_i \mathbf{L} \), where \( A_i = A_i(w, f, \mathbf{I}) \) is a square functional-entered matrix. Then

\[
A_i = \begin{pmatrix}
  cw(1-a^2w^2) & bw(1-b^2w^2) & -w & I \\
  dw(1-a^2w^2) & abdw^3 & w^3(1-b^2w^2) & I_x \\
  ew(1-a^2w^2) & abdw^3 & w^3(1-b^2w^2) & I_y \\
  \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14}
\end{pmatrix}
\]

\[
\rho_{11} = w^3 \left( 3a^2c^2w^2 + 6a^2bw^2 - 2c^2(1-3b^2w^2) + 3c^2 - 2abcd \right)
\]

\[
\rho_{12} = w^3 \left( be^2(1-3b^2w^2) + 6a^2bw^2 - 2acd + 3b^2d^2w^2 - 3bd^2 \right)
\]

\[
\rho_{13} = w^3 \left( c^2(1-3a^2w^2) - 6abcdw^2 + d^2(1-3b^2w^2) \right)
\]

\[
\rho_{14} = I_{xx}
\]

\[
\rho_{21} = w^3 \left( 3a^2d^2w^2 + 6a^2bdw^2 - 2a(e^2(1-3b^2w^2) + 3d^2) - 2bde \right)
\]

\[
\rho_{22} = w^3 \left( bd^2(1-3a^2w^2) + 6a^2bdw^2 - 2acd + 3b^2e^2w^2 - 3be^2 \right)
\]

\[
\rho_{23} = w^3 \left( 3a^2d^2w^2 + 6a^2bdw^2 - 2a(e^2(1-3b^2w^2) + 3d^2) - 2bde \right)
\]

\[
\rho_{31} = w^3 \left( 3a^2cdw^2 + 3a^2bw^2(ce + d^2) - ad(-3b^2ew^2 + 3e + c) - b(ce + d^2) \right)
\]

\[
\rho_{32} = w^3 \left( -bd(-3a^2ew^2 + c + 3e) + 3ab^2w^2(ce + d^2) \right)
\]

\[
\rho_{33} = w^3 \left( d(-3a^2ew^2 + 3a^2bw^2 + d) + d3ab^2w^2(ce + d^2) \right)
\]

\[
\rho_{34} = I_{yy}
\]

\[
\rho_{24} = I_{yy}
\]

\[
\rho_{34} = I_{xy}.
\]
Notice that for each \( i \), \( \det A_i = w^3(d^2 - ce)C_i \), and that \( s = 0 \implies w \neq 0 \). Suppose \( \neg(C_i = 0 \ \forall i) \). That is, \( \exists i : C_i \neq 0 \). Due to the non-degeneracy assumption and the constraint imposed by \( s \) that \( w \neq 0 \), this is equivalent to \( \exists i : w^3(d^2 - ce)C_i = \det A_i \neq 0 \iff \exists i : \ker(A_i) = \{0\} \). This is equivalent to \( \exists i : \forall L \neq 0, A_i L = r_i \neq 0 \), which is to say that there exists an \( i \) which, regardless of \( L \), will always violate one of the Lambert partial derivative conditions. This implies \( \neg((f, I) \text{ are consistent}) \). The contrapositive of this argument is that under the stated assumptions, \( (f, I) \) consistent implies that \( C_i = 0 \ \forall i \).

3. Derivation of \( T_I, S_I, \) and \( R_I \) from §4.3.

\[
T(t) := \begin{bmatrix} \cos(t) & -\sin(t) \\
\sin(t) & \cos(t) \end{bmatrix}, \tag{5}
\]

\[
S(t) := \begin{bmatrix} \cos^2(t) & \sin(2t) & \sin^2(t) \\
-\sin(t) \cos(t) & \cos^2(t) - \sin^2(t) & \sin(t) \cos(t) \\
\sin^2(t) & -2 \sin(t) \cos(t) & \cos^2(t) \end{bmatrix}, \tag{6}
\]

\[
R(t) := \begin{bmatrix} 1 & 0 & 0 \\
0 & T_I & 0 \\
0 & 0 & S_I^{-1} \end{bmatrix}, \tag{7}
\]

for which one can verify that \( f \in F(I) \) if and only if \( \hat{R}(t)f \in F(R(t)I) \), with \( \hat{R}(t) \) the principal submatrix of \( R(t) \) obtained by removing its first row and column. We obtain \( T_I, S_I, R_I \) from setting \( t = -\frac{i \log((t_x - T_x) - i \log(T_x^2 + T_y^2))}{\sqrt{T_x^2 + T_y^2}}. \)